THERMOCAPILLARY MOTION OF A DROP UNDER THE ACTION OF RADIATION

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Using the Stokes approximation and assuming smallness of the thermal Peclet number, an approximate analytical solution is found to the problem of velocity and temperature distribution in thermocapillary motion of a drop due to radiation absorbed on its surface in the absence of gravity. The drift rate and correction to the spherical drop surface form are found. It is shown that the results obtained are applicable to the case of arbitrary surface heat sources located symmetrically about an axis passing through the center of mass of the drop. Because of temperature dependence of the surface tension coefficient the presence of an inhomogeneous temperature distribution over the drop surface leads to discontinuities in tangential stress on the surface, which produces various thermocapillary effects such as instability of the rest state and drift at constant velocity in the absence of gravity [1-4]. The literature has considered various mechanisms of development of an inhomogeneous surface temperature distribution. In one case it was related to the asymmetric distribution of the heat source, independent of drop motion [1, 2], while in another case in the rest state the surface was heated uniformly and a temperature gradient developed only upon motion of the drop, whereupon change in surface tension then affected the motion [3, 4]. The present study will investigate the first case of thermocapillary motion of a drop of viscous liquid located in another viscous liquid with which it will not mix, the latter liquid filling all space, upon irradiation of the drop from one direction by a planoparallel light beam with homogeneous cross section in the absence of gravity. It will be assumed that the radiation is absorbed totally on the drop surface and that the surrounding medium is transparent. We will consider the established slow motion of the drop along the direction of the incident radiation. We use the Stokes equation and neglect convective terms in the thermal conductivity equation [5]. We assume that the drop surface maintains a spherical form. The density, viscosity, thermal conductivity, and specific heat of the liquids inside and outside the drop will be assumed constant, while the surface tension coefficient is a linear function of temperature.

We will use a reference frame fixed to the center of the drop, in which the problem reduces to planoparallel liquid flow about the drop. Within the framework of the assumptions made, the equations and boundary conditions for the flow function and temperature can be written in the form

$$E^4\psi_i = 0,\ E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}; \tag{1}$$

$$\begin{gathered} r \to \infty,\ \psi_1 \to r^2(1-\mu^2)/2,\ r = 0,\ \psi_2/r^3 < \infty, \\ r = 1,\ \psi_1 = \psi_2 = 0,\ \partial\psi_1/\partial r = \partial\psi_2/\partial r; \end{gathered} \tag{2}$$

$$\left(2\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2}\right)(\psi_1 - \beta\psi_2) = \mathrm{Ma}\,(1-\mu^2)\frac{\partial \varphi_1}{\partial \mu}; \tag{3}$$

$$\Delta\varphi_i = 0; \tag{4}$$

$$\begin{gathered} r \to \infty,\ \varphi_1 \to 0,\ r = 0,\ \varphi_2 < \infty, \\ r = 1,\ \varphi_1 = \varphi_2,\ \partial\varphi_1/\partial r - \delta\partial\varphi_2/\partial r + f(\mu) = 0; \end{gathered} \tag{5}$$

$$f(\mu) = \begin{cases} -\mu, & -1 \leqslant \mu \leqslant 0, \\ 0, & 0 < \mu \leqslant 1, \end{cases} \tag{6}$$

$$\mu = \cos\theta,\ v_{ir} = \frac{1}{r^2\sin\theta}\frac{\partial\Psi_i}{\partial\theta},\quad v_{i\theta} = \frac{-1}{r\sin\theta}\frac{\partial\Psi_i}{\partial r},$$

$$\beta = \frac{\mu_2}{\mu_1},\ \delta = \frac{\lambda_2}{\lambda_1},\ \psi_i = \frac{\Psi_i}{U_\infty a^2},\ \varphi_i = \frac{\lambda_1(T_i - T_\infty)}{Ia},\ \mathrm{Ma} = \frac{Ia}{\mu_1\lambda_1 U_\infty}\frac{d\sigma}{dT}.$$

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Here and below the subscripts i = 1, 2 refer to the external medium and the drop, respectively; $U_\infty$ is the velocity of the incident flow which is determined by the condition that the force acting on the drop vanish ($U_\infty > 0$ if the velocity is directed along the x axis, $U_\infty < 0$ in the opposite case); $\Psi_i$, $v_i$, and $T_i$ are the flow function, velocity, and temperature; $\mu_i$, $\lambda_i$, and $\sigma$ are the dynamic viscosity, thermal conductivity, and surface tension coefficients; $T_\infty$ is the temperature far from the drop; $a$ is the drop radius which is used as a length scale in dedimensionalizing; $f(\mu)$ is the surface heat liberation function; I is the intensity of the incident radiation; and Ma is the Marangoni number. The x axis is directed in the direction of radiation propagation and passes through the center of the drop.

A spherical coordinate system will be used, in which the dimensionless radius r is measured from the center of the drop, and the angle $\theta$ is measured from the positive x axis.

According to [6], the solution of problem (1) with conditions (2) has the form

$$\psi_1 = \left(r^2 + Ar - \frac{A+1}{r}\right)\frac{1-\mu^2}{2} + \sum_{n=3}^{\infty} A_n (r^{-n+3} - r^{-n+1}) G_n(\mu),$$
$$\psi_2 = \left(A + \frac{3}{2}\right)(r^4 - r^2)\frac{1-\mu^2}{2} + \sum_{n=3}^{\infty} A_n (r^{n+2} - r^n) G_n(\mu) \qquad (7)$$

[$G_n(\mu)$ is a Gegenbauer function of the first sort of order n and degree −1/2]. The constants A, $A_n$ (n = 3, 4, ...) are yet to be defined, and are found with Eq. (3) after solving the temperature distribution problem.

The solution of Eq. (4) with boundary conditions (5) is

$$\varphi_1 = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\mu), \quad \varphi_2 = \sum_{n=0}^{\infty} B_n r^n P_n(\mu); \qquad (8)$$

$$B_n = \frac{1}{2}\,\frac{2n+1}{n(\delta+1)+1}\int_{-1}^{1} f(\mu) P_n(\mu)\,d\mu, \quad n = 0, 1, 2 \ldots \qquad (9)$$

[$P_n(\mu)$ are n-th order Legendre polynomials of the first sort].

Substituting Eqs. (7) and (8) in Eq. (3), we find

$$A = \left[\frac{\mathrm{Ma}\,B_1}{3} - \left(1 + \frac{3}{2}\beta\right)\right](1+\beta)^{-1},$$
$$A_n = \frac{\mathrm{Ma}\,B_{n-1} n (n-1)}{2(2n-1)(1+\beta)}, \quad n = 3, 4 \ldots \qquad (10)$$

To complete the solution of the problem of flow over the drop, the velocity $U_\infty$ or the drift velocity $U_*$ ($U_* = -U_\infty$) must be found.

According to [6], the force acting on the drop is

$$F = -4\pi\mu_1 a A U_\infty. \qquad (11)$$

If F > 0, the force is directed along the x axis, while if F < 0 it is opposite the x axis. After substitution of Eq. (10), Eq. (11) can be written as

$$F = 4\pi\mu_1 a\left[\left(1 + \frac{3}{2}\beta\right)U_\infty - \frac{d\sigma}{dT}\,\frac{IaB_1}{3\mu_1\lambda_1}\right](1+\beta)^{-1}. \qquad (12)$$

In the absence of gravitation the drift velocity $U_*$ can be found from the condition that the force of Eq. (12) vanish:

$$U_* = -\frac{d\sigma}{dT}\,\frac{IaB_1}{3\mu_1\lambda_1}\left(1 + \frac{3}{2}\beta\right)^{-1}. \qquad (13)$$

It should be noted that both the solution of Eq. (7) for flow over the drop with consideration of Eq. (10), and Eq. (12) for the force acting on the drop can be written as the sum of two terms, one of which defines conventional Stokes flow over the drop at a velocity $U_\infty$ and the conventional expression for the force acting on the drop [6], while the second represents purely thermocapillary motion and a purely thermocapillary force.

Equations (7)-(13) are applicable to an arbitrary distribution of surface heat sources symmetric about an axis passing through the center of the drop. Then $f(\mu)$ will be a dimensionless surface heat source distribution function and I will be the characteristic heat liberation on the surface. It can easily be seen that if, in addition, the heat sources are arranged symmetrically relative to a plane passing through the drop center perpendicular to the axis of symmetry, then the thermocapillary stresses produced by such sources do not affect the drop motion in the approximation considered ($B_1$ vanishes), although generally speaking the flow inside and outside the drop does change [not all $B_n$ ($n = 2, 3, \ldots$) vanish]. To find the effect of such sources on drop motion it is necessary to consider the next term in the expansion of the temperature in terms of the small Peclet number, as was done in [3].

If the heat source is radiation absorbed on the surface, then from Eqs. (6) and (9) we obtain $B_1 = -(\delta + 2)^{-1}/2$ and the expression for the drift velocity in the absence of gravitation can be rewritten as

$$U_* = \frac{d\sigma}{dT}\,\frac{Ia}{6\mu_1\lambda_1(\delta+2)}\left(1+\frac{3}{2}\beta\right)^{-1}. \tag{14}$$

Since for the majority of substances $d\sigma/dT < 0$, as follows from Eq. (14), the drop will drift toward the beam. Qualitative considerations affect the stability of thermocapillary drop drift. In fact, when the drift velocity deviates from the equilibrium value an additional force appears to oppose this change.

The conditions of smallness of the Reynolds and Peclet numbers used in the present study impose certain limitations on the parameter values for which the solution can be considered correct. In formulating the problem of Eqs. (1)-(5) the boundary condition for normal stresses on the drop surface

$$r = 1,\quad -\frac{\mathrm{We}}{\mathrm{Re}}(p_1 - \beta p_2) - 2\frac{\mathrm{We}}{\mathrm{Re}}(\partial^2_{r\mu}\psi_1 - \beta\partial^2_{r\mu}\psi_2) = 2h\left(1 + \frac{\mathrm{We}}{\mathrm{Re}}\mathrm{Ma}\,\varphi_1\right), \tag{15}$$
$$\mathrm{We} = \rho_1 a U_\infty^2/\sigma_\infty,\quad \mathrm{Re} = \rho_1 a U_\infty/\mu_1$$

was omitted. Here $p_1$ and $p_2$ are the pressures outside and inside the drop, referenced to $\mu_1 a^{-1}U_\infty$ and $\mu_2 a^{-1}U_\infty$, respectively; $\sigma_\infty$ is the surface tension coefficient at the temperature far from the drop; We and Re are the Weber and Reynolds numbers; $h = Ha/2$; H is the curvature of the drop surface (for a spherical drop $H = 2/a$, $h = 1$); $\rho_i$ is the density.

By substituting the known solutions (7) and (8) with consideration of Eqs. (9) and (10) and the expressions for the pressure which can easily be found, then knowing the flow function [6], one can easily prove that, generally speaking, Eq. (15) is not satisfied. This means that the drop form cannot remain spherical ($h \neq 1$). However, upon satisfaction of the condition $\varepsilon = \mathrm{Ma}, \mathrm{We}/\mathrm{Re} \ll 1$, the deviation from spherical form will be small, and Eq. (15) must then be considered as a boundary condition for normal stresses on the spherical surface ($r = 1$), which in the main approximation reduces to a Laplace pressure discontinuity on the drop surface.

We will seek the surface shape in the form

$$R(\mu) = 1 + \varepsilon\zeta(\mu); \tag{16}$$

$$\zeta(\mu) = \sum_{n=2}^{\infty} \alpha_n P_n(\mu). \tag{17}$$

The expansion of Eq. (17) begins with the term with number $n = 2$, since upon deformation of the surface the drop volume does not change and the origin of the coordinate system was chosen at the center of mass.

The dimensionless curvature h can be represented by an expansion in a small parameter

$$h = 1 + \varepsilon h^{(1)} \tag{18}$$

and, in view of the relationship $h^{(1)} = -\zeta - (1/2)d((1 - \mu^2)d\zeta/d\mu)/d\mu$, we will have

$$h^{(1)} = \sum_{n=2}^{\infty} \gamma_n P_n(\mu), \quad \gamma_n = \alpha_n (n-1)(n+2)/2. \tag{19}$$

Substituting in Eq. (15) Eqs. (7), (8), and (18) with consideration of Eqs. (9), (10), (19), and the expressions for the pressure [6], we obtain

$$A = 0; \tag{20}$$

$$\alpha_n = \frac{\beta - n\beta - n - 2}{(2n+1)(n-1)(n+2)(1+\beta)} B_n, \; n = 2, 3 \ldots \tag{21}$$

Equation (20) reflects the fact that the force acting on the drop is equal to zero, while Eq. (21), together with Eqs. (16) and (17), defines the surface form.

If we assume that a gravitational force exists, parallel to the direction of radiation propagation (or to the axis of symmetry of the heat sources), then the velocity of drop motion will no longer satisfy Eqs. (13) or (14), but can be found from the condition that the resultant force acting on the drop, equal to the sum of the force of Eq. (12) and the mass force, vanish. The intensity of the radiation required to maintain the drop at rest can be determined from the same condition. As before, Eqs. (16), (17), and (21) are valid, since, as follows therefrom, in the approximation considered the drop form does not depend on the velocity $U_\infty$, and, in the final reckoning, on the force of gravity. This is of course not surprising, since the form is determined solely by the pure thermocapillary terms of the flow function.

At Ma = 0 or $\delta \to \infty$ (high thermal conductivity of the drop material) the temperature is constant over the surface and the thermocapillary effect disappears. Then from Eq. (7) we obtain the Rybchinskii–Adamar solution, while Eq. (12) yields the conventional expression for the drop resistance force [6], the form remains precisely spherical, and the drift velocity given by Eqs. (13) or (14) vanishes.

As $\beta \to \infty$ (high viscosity of drop material) motion within the drop is frozen, thermocapillary stresses play no role at all, and Eqs. (7) and (12) reduce to the corresponding expressions for Stokes flow over a rigid sphere and the Stokes force [6]. The drift velocity vanishes. However, the drop form remains nonspherical. This is true because although the liquid motion within the drop is very weak, in view of the high viscosity that motion does lead to marked stresses and pressure changes which cause the nonspherical form, as does change in the surface tension coefficient over the surface.

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